Political Model of Social Evolution

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Supplementary Material

1 Proofs

We first formulate and prove the following auxiliary result.

Lemma 1 *Suppose that $\phi : S \to S$ such that $|\phi(s) - s| \leq 1$ for any $s \in S$ is a transition mapping, and payoff vectors $\{u_i(s)\}_{i \in N}^{s \in S}$ satisfy Strict Increasing Differences (i.e., $u_i(s) - u_j(s)$ is strictly increasing in s for $i > j$ or, equivalently, $u_i(x) - u_i(s)$ is strictly increasing in i for $x > s$). Then for any $0 \leq \beta < 1$, $\{V_i(s)\}_{i \in N}^{s \in S}$, where*

$$V_i(s) = u_i(s) + \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(s)\right), \tag{1}$$

satisfy Strict Increasing Differences property, too.

Proof of Lemma 1. We prove this result by induction by the number $\mu(\phi)$ of "non-monotonicities", i.e., the number of pairs of states (x, y) such that $x < y$, but $\phi(x) > \phi(y)$.

Base: Suppose first that this number $\mu(\phi) = 0$, i.e., $\phi(s)$ is monotonic. Let We have

$$\begin{aligned} V_i(y) - V_i(x) &= u_i(y) + \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(y)\right) - u_i(x) - \sum\nolimits_{k=1}^{\infty} \beta^k u_i\left(\phi^k(x)\right) \\ &= (u_i(y) - u_i(x)) + \sum\nolimits_{k=1}^{\infty} \beta^k \left(u_i\left(\phi^k(y)\right) - u_i\left(\phi^k(x)\right)\right). \end{aligned} \tag{2}$$

The first term is strictly increasing in i, and the rest are weakly increasing in i, as $\phi^k(y) \geq \phi^k(x)$ for $k \geq 1$ due to monotonicity. Consequently, (2) is strictly increasing in i, and this proves the base.

Step: Take a pair of states (x, y) such that $x < y$, but $\phi(x) > \phi(y)$. We must have $y = x+1$. Define

$$\tilde{u}_i(s) = \begin{cases} \frac{u_i(x)+\beta u_i(y)}{1+\beta} & \text{if } s = x, \\ \frac{u_i(y)+\beta u_i(x)}{1+\beta} & \text{if } s = y, \\ u_i(s) & \text{otherwise.} \end{cases} \tag{3}$$

Then $\{\tilde{u}_i(s)\}_{i\in N}^{s\in S}$ satisfy SID as well. Indeed, take any $a<b$; if $(a,b)\neq(x,y)$, then the property holds trivially. Otherwise, suppose $a<x=b<y$; we then have

$$\begin{aligned}\tilde{u}_i(b)-\tilde{u}_i(a) &= \frac{u_i(x)+\beta u_i(y)}{1+\beta}-u_i(a)\\ &= \frac{u_i(x)-u_i(a)}{1+\beta}+\beta\frac{u_i(y)-u_i(a)}{1+\beta}. \end{aligned} \tag{4}$$

In (4), the first term is strictly increasing, and the second is weakly increasing (strictly if $\beta>0$) in i. We can consider cases $a<x<y=b$, $a=x<y<b$, and $x<a=y<b$ similarly. Finally, if $a=x<y=b$, we have

$$\begin{aligned}\tilde{u}_i(b)-\tilde{u}_i(a) &= \frac{u_i(y)+\beta u_i(x)}{1+\beta}-\frac{u_i(x)+\beta u_i(y)}{1+\beta}\\ &= \frac{1-\beta}{1+\beta}\left(u_i(y)-u_i(x)\right), \end{aligned} \tag{5}$$

which is also strictly increasing in i.

Define mapping $\tilde{\phi}:S\to S$ by

$$\tilde{\phi}(s)=\begin{cases} y & \text{if } s=x,\\ x & \text{if } s=y,\\ \phi(s) & \text{otherwise.}\end{cases} \tag{6}$$

Clearly, $\mu\left(\tilde{\phi}\right)=\mu(\phi)-1$, and by induction $\left\{\tilde{V}_i(s)\right\}_{i\in N}^{s\in S}$, given by

$$\tilde{V}_i(s)=\tilde{u}_i(s)+\sum\nolimits_{k=1}^{\infty}\beta^k\tilde{u}_i\left(\tilde{\phi}^k(s)\right),$$

satisfy SID. But for any $i\in N$ and $s\in S$, $\tilde{V}_i(s)=V_i(s)$ (this trivially holds for $s=x$ and $s=y$, and one can use induction by $|s-x|$ for $s<x$ and by $|s-y|$ for $s>y$ to verify the equality). Consequently, $\{V_i(s)\}_{i\in N}^{s\in S}$ satisfy SID, and which completes the induction step. ■

Proof of Proposition 1. We prove a stronger result, obtained by weakening one condition in Part 3. We replace the assumption of single-peakedness with the following requirement: "For each state s there is a player $i(s)\in M_s$ such that there do not exist two states $x<s$ and $y>s$ such that $u_i(x)>u_i(s)$ and $u_i(y)>u_i(s)$". This is trivially satisfied if for each player i, preferences are single-peaked (then one can pick $i(s)$ to be any element of M_s.

We start by proving this stronger proposition in the non-stochastic case, i.e., where $L_0=S$. We do so by induction by the number of states $m=|S|$. The base $m=1$ is trivial. We now assume that the Proposition has been proved for all configurations with $|S|$ less than m, and now prove the induction step for each part of the Proposition.

Proof for the non-stochastic case. Part 1. Consider two possibilities.

Possibility 1. Suppose first that $u_i(1) \le u_i(2)$ for at least one $i \in M_2$. We consider a new game with the same set of players N, the same set of states $S' = [2, m]$, the same set of winning coalitions on S' and the same protocols on S', and payoffs given by $\tilde{u}_i(x) = u_i(x)$ for each $x \in S'$. For this new game, Assumption 1 holds, and by induction it has MPE σ' with transition mapping ϕ'. Let $\{V_i'(x)\}$ be the continuation values in this MPE. Let us define $\phi : S \to S$ by setting $\phi(x) = \phi'(x)$ if $x \in S'$, and $\phi(1) = 1$ if $V_i'(2) \le u_i(1) / (1-\beta)$ for some $i \in M_1$ and $\phi(1) = 2$ otherwise. Let us construct MPE σ that implements ϕ.

We need to define proposing and voting strategies for each state $s \in S$. If state $s \in S'$, we require players to make proposals $a_{t,i}$ as they did in the game with the set of states S'. If proposal $a_{t,i} \in S'$ is made, players vote as in σ'; while in addition, if proposal $a_{t,i} = 1$ is made, we require that all players who are have $u_i(1) \le u_i(2)$ vote against this proposal (the players who support the proposal will therefore not make a winning coalition in state 2). Finally, in state 1, we consider two cases. If we defined $\phi(1) = 1$, then we can require that only proposal $a_{t,i} = 1$ is made, and proposal 2 (in fact, any proposal), if made, is rejected. If we defined $\phi(2) = 2$, we require that proposal 2 is accepted and proposal 1 is rejected (we do so by making all players i with $V_i'(2) > u_i(1) / (1-\beta)$ support proposal 1); in addition, we require that all players i with $V_i'(2) > u_i(1) / (1-\beta)$ propose alternative 2, while the rest propose 1. Let us verify that strategies constructed in this way (profile σ) are MPE; to do so, notice that continuation values on $V_i(s) = V_i'(s)$ for all $s \in S'$.

First, suppose the current state is $s \in S'$. Then voting strategies if proposal $a_{t,i} \in S'$ is made are best responses for each player, as continuation game yields the same payoffs. Suppose state 1 is proposed. If it is rejected, there are two possibilities: that a transition to state 3 occurs or that the society remains in state 2; the latter is possible only if $\{V_i'(3) \ge V_i'(2)\}$ for a winning coalition in state 2. Since $V_i'(2) \le u_i(1) / (1-\beta)$ for some $i \in M_1$, we must have $V_i(2) \le V_i(1)$ for this i, and thus proposal 1 will be rejected if doing so will lead to staying in 2. If, however, rejecting proposal 1 will lead to 3, this means that $V_j'(3) \ge V_j'(2)$ for all $j \in M_2$, and to $j = i$ in particular. This again means that rejecting 1 is best response for sufficiently many players (for all players $k \ge j$). In light of this, proposing 1 only makes sense for a player who would want to stay in 2 instead of moving to 3. However, this player would achieve the same by proposing 2, and thus proposing 1 cannot be a profitable deviation from proposing 2, and if it is a profitable deviation from proposing 3, then proposing 2 must have been a profitable deviation from proposing 3 in profile σ', which is impossible. This shows that making the same proposals as in σ' is best response for each player at each stage.

Now consider state 1. If $\phi(1) = 1$, then a proposal to move to 2 may be required never to be accepted, as $V_i'(2) \le u_i(1) / (1-\beta)$ for sufficiently many players, and staying in 1 would yield continuation utility $V_i(1) = u_i(1)/(1-\beta)$ while moving to 2 would yield $V_i(2) = V_i'(2)$. Therefore, making any proposal is a best response. Suppose, finally, that $\phi(1) = 2$. In that case, supporting proposal 2 is a best response for a winning coalition of players in state 1; at the same time, rejecting proposal 1 is also a best response for these players. This means that for any individual $i \in M_1$, proposing 2 is a best response. We proved that profile σ is a Markovian profile consisting of best responses, i.e., a MPE.

Possibility 2. The second possibility is that $u_i(1) > u_i(2)$ for all $i \in M_2$. We take a new game with the set of states $S' = [2, m]$, same sets of winning coalitions and same protocols on S', but with payoffs given by $\tilde{u}_i(x) = u_i(x)$ for each $x \geq 3$ and $\tilde{u}_i(x) = (1-\beta)u_i(2) + \beta u_i(1)$ if $x = 2$. Again, Assumption 1 holds, so we can take MPE σ' with transition mapping ϕ'; denote the continuation values by $\{V_i'(x)\}$. We then define $\phi(x) = \phi'(x)$ for all $x \geq 3$ and consider the following cases separately.

Case 1. First, if at least one $i \in M_2$ has $u_i(2)/(1-\beta) \geq V_i'(3)$, then we let $\phi(2) = \phi(1) = 1$. Then MPE σ may be constructed as follows. For $s \geq 3$, we can take strategies from σ'. For $s = 2$, we require that proposal 3 be rejected if made (and so will proposal 2), while proposal 1 be accepted; in addition we require that players with $u_i(1) > u_i(2)$ actually propose 1. For $s = 1$, we can require that all proposals are rejected. As in the consideration of Possibility 1, we can prove that σ chosen in this way is a MPE.

Case 2. Second, suppose that all players $i \in M_2$ have $V_i'(3) > u_i(2)/(1-\beta)$. Take the player (not necessarily in M_2) for whom these two inequalities are satisfied and who is the last to propose when the state is 2; denote this player by j. If either $u_i(1)/(1-\beta) \geq V_i'(3)$ for all $i \in M_2$, or this is true for at least one player in M_2 and for player j, then let $\phi(2) = \phi(1) = 1$. The corresponding σ is similar to Case 1, except that if the current state is 2, we have to consider the two situations separately. If $u_i(1)/(1-\beta) \geq V_i'(3)$ for all $i \in M_2$, then we require that proposals 2 and 3 are never accepted, while proposal 1 is accepted, and actually made by any player with $u_i(2)/(1-\beta) \geq V_i'(3)$. If $u_i(1)/(1-\beta) \geq V_i'(3)$ for at least one player in M_2 and for player j, then we require that j propose 1 and it is accepted, but if j proposed 3, it would also be accepted; in addition, any proposal made by any other player is not accepted. It is straightforward to prove that σ constructed in such a way is a MPE that implements ϕ.

Case 3. The remaining case is where all players $i \in M_2$ have $V_i'(3) > u_i(2)/(1-\beta)$, but $u_i(1)/(1-\beta) < V_i'(3)$ for at least one $i \in M_2$ and, moreover, either this holds for all players in M_2 or for player j. We let $\phi(2) = 3$. Now, let $\phi(1) = 2$ if $(1-\beta)u_i(2) + \beta V_i'(3) > u_i(1)$ for all $i \in M_1$, and $\phi(1) = 1$ otherwise. We then construct σ in the following way. First, we notice that in this case, we must have $\phi'(3) \geq 3$ (and thus $\phi(3) \geq 3$): indeed, $\phi'(3) = 2$ is impossible, since $V_i'(3) > \tilde{u}_i(2)/(1-\beta)$ for at least one $i \in M_2$, and thus, by Assumption 1, this inequality must hold for at least one $i \in M_3$. This implies that for $s \geq 3$, $V_i(s) = V_i'(s)$ for all i.

So, for states $s \geq 3$, we take strategies from σ'; in that case, state 2 will not be accepted if proposed. For state 2, we consider two cases. If $u_i(1)/(1-\beta) < V_i'(3)$ for all $i \in M_2$, then we require that proposals 1 and 2 never be accepted, while proposal 3 be accepted and, moreover, any player with $V_i'(3) > u_i(2)/(1-\beta)$ proposes 3. If $u_i(1)/(1-\beta) < V_i'(3)$ for player j defined above, we require that he proposes state 3 when it is his turn, and players accept; though, if he made proposal 1, it may also be accepted. Any earlier proposal is rejected. Again, it is straightforward to check that we can finish the construction of σ such that σ is MPE, and, moreover, it implements ϕ.

This completes the proof of existence in the non-stochastic case.

Part 2. Generically, $u_i(x) \neq u_i(y)$ for any $x, y \in S$ and $i \in N$. Suppose, to obtain a contradiction, that mapping ϕ supported by MPE σ is nonmonotonic. Then there are $x, y \in S$ such $x < y$ and $\phi(x) > \phi(y)$. Since transitions are one-step, we have $y = x + 1$.

Let us prove that $u_i(y) > u_i(x)$ for all $i \in M_x$. For every $i \in S$, $V_i(x) = u_i(x) + \beta V_i(y)$ and $V_i(y) = u_i(y) + \beta V_i(x)$, which implies $V_i(x) - V_i(y) = (1+\beta)(u_i(x) - u_i(y))$.

Let us prove that $V_i(y) \geq V_i(x)$ for all $i \in M_x$. Suppose not; then consider the last voting in the protocol π^x for which proposal $z \in \{x-1, x, y\}$ such that $V_j(z) < V_j(x)$ for some $j \in M_x$ is accepted. By construction, if z is rejected at this stage, then alternative s, which has $V_j(s) > V_j(z)$, will be implemented, which means that j is better off rejecting z. But by Lemma 1, $\{V_i(s)\}_{i \in N}^{s \in S}$ satisfies strict increasing differences, which means that then a winning coalition is not better off accepting z. This contradicts that z is accepted at this stage.

We thus proved that $V_i(x) \geq V_i(y)$. Given genericity, $u_i(y) > u_i(x)$ for all $i \in M_x$. We can similarly prove that $u_i(x) > u_i(y)$ for all $i \in M_y$. But by Assumptions 1 and 3, the same must hold for every $i \in M_x$, which contradicts the opposite inequality established in the previous paragraph. This contradiction completes the proof.

Part 3. Suppose that there are two MPEs σ_1 and σ_2 and two different transition mappings ϕ_1 and ϕ_2 corresponding to these MPEs, respectively. Without loss of generality, assume that m is the minimal number of states for which this is possible, i.e., if $|S| < m$, then transition mapping is unique. Obviously, $m \geq 2$.

Let us first prove that if $\phi_1(x) = x$, then $x = 1$ or $x = m$. Indeed, suppose the opposite, and consider $\phi_2(x)$. If $\phi_2(x) < x$, then $\phi_1|_{[1,x]}$ and $\phi_2|_{[1,x]}$ are two different mappings, both of which may, as it is easy to show, be transition mappings for MPE in the game with the same players but with the set of states $S' = [1, x]$. This would contradict the assertion that m is the minimal number of players for which this is possible. If $\phi_2(x) > x$, we get a similar contradiction by considering the subset of states $[x, m]$, and if $\phi_2(x) = x$, we get a contradiction by considering

$[1, x]$ or $[x, m]$ depending on where ϕ_1 and ϕ_2 differ. We similarly prove that if $\phi_2(x) = x$, then either $x = 1$ or $x = m$.

We now consider the two cases of the Proposition separately.

(a) Generically, no player gets the same utilities in two different states, and both mappings are monotone. If $\phi_1(x) < x < \phi_2(x)$ or vice versa, then for all $i \in M_x$, there must be both a state $x_1 < x$ and a state $x_2 > x$ such that $u_i(x_1) > u_i(x)$ and $u_i(x_2) > u_i(x)$, which contradicts the assumption in this case. Since for $1 < x < m$, $\phi(x) \neq x$, we get that $\phi_1(x) = \phi_2(x)$ for such x. Let us prove that $\phi_1(1) = \phi_2(1)$. If this is not the case, then $\phi_1(1) = 1$ and $\phi_2(1) = 2$ (or vice versa). If $m = 2$, then monotonicity implies $\phi_2(2) = 2$, and if $m > 2$, then, as proved earlier, we have $\phi_2(x) = x + 1$ for $1 < x < m$ and $\phi_2(m) = m$. In both cases, we have $\phi_1(x) = \phi_2(x) > 1$ for $1 < x \leq m$. Hence, $V_i^1(2) = V_i^2(2)$ for all $i \in N$ (where V^1 and V^2 are continuation payoffs under ϕ_1 and ϕ_2, respectively). Since σ_1 is MPE, we must have $u_i(1) / (1 - \beta) \geq V_i^1(2)$ for $i \in M_1$, and since σ_2 is MPE, we must have $u_i(1) / (1 - \beta) \leq V_i^2(2)$. Generically, this cannot hold, and this proves that $\phi_1(1) = \phi_2(1)$. We can similarly prove that $\phi_1(m) = \phi_2(m)$, which implies that $\phi_1 = \phi_2$. This contradicts the hypothesis of non-uniqueness.

(b) In this case, let M_x denote the unique quasi-median voter in state $x \in S$, and let $b(x)$ be the state that maximizes $u_{M_x}(y)$ on S (generically, it is unique). By Assumption 1 the sequence $\{b(x)\}_{x=1}^m$ is nondecreasing. Let us prove that $b(2) \geq 2$. Indeed, if $b(2) = 1$, then $b(1) = 1$ by monotonicity, and hence we must have $\phi_1(1) = \phi_2(1) = 1$, and therefore $\phi_1(2) = \phi_2(2) = 1$. Now consider a game with the same set of players, set of states $S' = [2, m]$, same sets of winning coalitions, and payoffs given by $\tilde{u}_i(x) = u_i(x)$ for $x > 2$, $\tilde{u}_i(2) = (1 - \beta) u_i(2) + \beta u_i(1)$. Now, notice that perpetual state 2 in the new game delivers exactly the continuation utility $V_i^1(2) = V_i^2(2)$ of the original game. It is now easy to see that the two mappings $\tilde{\phi}_1$ and $\tilde{\phi}_2$ given by $\tilde{\phi}_j(x) = \phi_j(x)$ if $x > 2$, $\tilde{\phi}_j(2) = 2$, may be supported by MPE in the new game. But these are different mappings, which contradicts that m is the minimal number of players for which this is possible. Hence, $b(2) \geq 2$. We can similarly prove that $b(m - 1) \leq m - 1$. Since $\{b(x)\}_{x=2}^{m-1}$ is nondecreasing, $b(x) = x$ for some $x \in [2, m - 1]$. But this would imply that $\phi_1(x) = x$, which we earlier proved to be impossible. This contradiction completes the proof in the nonstochastic case.

Proof for the stochastic case. Part 1. The proof proceeds by induction by the number of state which are in S but not in L. The base was proved earlier. for the step, we introduce some extra notation. For any set of available states L, let $L^- = L \cup \{\min L - 1\}$, $L^+ = L \cup \{\max L + 1\}$. Let us denote the transition mapping if the set of available states is L by ϕ^L, and let us denote the continuation utilities if the current set of available states is L, and a decision to transit to (or stay in) state s has just been made by $\left\{V_i^L(s)\right\}_{i \in N}^{s \in S}$. For consistency in formulas, let us denote

$V_i^L(s) = 0$ if L is not a subset of S (i.e., contains either 0 or $m+1$) for all $s \in [0, m+1]$; these will always be multiplied by 0 below.

Step. Suppose that we have proved the existence of MPE in pure strategies with a monotonic transition mapping for all proper supersets of L (which are subsets of S), and obtained continuation utilities. Let us now prove the same if the set of available states is L. We can write the continuation utility in the following way:

$$\begin{aligned} V_i^L(s) &= u_i(s) \\ &+\beta\left(p_L^l(1-p_L^r)V_i^{L^-}\left(\phi^{L^+}(s)\right) + \left(1-p_L^l\right)p_L^r V_i^{L^+}\left(\phi^{L^+}(s)\right) + p_L^l p_L^r V_i^{(L^-)^+}\left(\phi^{(L^-)^+}(s)\right)\right) \\ &+\beta\left(1-p_L^l\right)(1-p_L^r)V_i^L\left(\phi^L(s)\right). \end{aligned}$$

Now, if we denote

$$\begin{aligned} \tilde{u}_i(s) &= u_i(s) + \beta\left(p_L^l(1-p_L^r)V_i^{L^-}\left(\phi^{L^+}(s)\right) + \left(1-p_L^l\right)p_L^r V_i^{L^+}\left(\phi^{L^+}(s)\right) + p_L^l p_L^r V_i^{(L^-)^+}\left(\phi^{(L^-)^+}(s)\right)\right), \\ \tilde{\beta} &= \beta\left(1-p_L^l\right)(1-p_L^r), \end{aligned}$$

then we can write the continuation utility as

$$V_i^L(s) = \tilde{u}_i(s) + \tilde{\beta}V_i^L\left(\phi^L(s)\right).$$

Let σ^L be a MPE in pure strategies with monotonic transition map for a game without shocks, with utilities given by $\{\tilde{u}_i(s)\}_{i\in N}^{s\in S}$ and discount factor $\tilde{\beta}$ (its existence has been proved) The existence of such equilibrium has already been proved, as $\{\tilde{u}_i(s)\}_{i\in N}^{s\in S}$ satisfy the strict increasing differences condition, as may be easily shown. It is evident that the strategies taken from σ^L before any shock happens, combined with strategies found on the earlier stages of the induction played after a shock occurs, will constitute a MPE of the game. This MPE will be in pure strategies and with monotonic transition map. This proves the induction step and completes the proof of Part 1.

Part 2. This follows directly from Part 2 above (mapping $\tilde{\phi}$ is constucted as a transition mapping of some game without shocks).

Part 3. It suffices to prove that respective conditions hold in a game with the set of states L, utilities given by $\tilde{u}_i(s)$ and discount factor $\tilde{\beta}$. In case (b), it follows from the hypothesis for the game with set of states S. In case (a), take any state s, player $i(s)$ for whom the condition holds for utilities $u_i(s)$, and without loss of generality suppose $u_i(s) \geq u_i(x)$ for all $x > s$. If so, in the game with the set of states L', $\phi'(s) > s$ holds only if $V_i'(s) = u_i(s)/(1-\beta)$, in which case $V_i'(x) \leq V_i'(s)$, as the trajectory may take x only to states to the right of s. If $\phi'(s) \leq s$, then $V_i'(s) \geq u_i(s) / (1-\beta)$ (this holds for all $i \in M_s$, including $i(s)$). The trajectory starting from $x > s$ will either involve states that yield at most $u_i(s)$ per period, or will lead to state s, thus delivering a continuation value of $V_i'(s)$. This implies that $V_i'(x) \leq V_i'(s)$ for all x in this case as well. But this implies that $\tilde{u}_i(s) \geq \tilde{u}_i(x)$ for all $x > s$. This proves that the property in case (a) is satisfied, and Part 3 of the Proposition in the non-stochastic case is applicable. ■

1.1 Examples

Here, we show how our model may be used to illustrate some patterns in social evolution.

Example 1 ***(Early shocks may make reforms more difficult)*** Suppose that $S = \{s_1, s_2, s_3\}$ and we start with the set of feasible states, $L_0 = \{s_1, s_2\}$. Suppose also that s_2 corresponds to limited democracy and a change to this state will shift political power away from player 1. Let player 2 to be the quasi-median voter in s_2, i.e., $M_2 = \{2\}$. Finally, suppose that $u_2(s_3) > u_2(s_2)$ and $u_1(s_3) < u_1(s_2)$, so that player 2 would prefer transition towards a more democratic state s_3 if such transition were feasible, but this transition is disliked by player 1. For instance, we can think of player 1 as corresponding to the king or to the aristocracy and being in favor of limited democracy, but disliking full democracy (see Acemoglu, Egorov and Simon 2010a, for a more detailed discussion of this example). Suppose that β is sufficiently high so that when $L = \{s_1, s_2, s_3\}$, player 1 prefers to maintain s_1 (since a change to s_2 will immediately induce a change to s_3, which he dislikes).

Yet when we start with $L_0 = \{s_1, s_2\}$ and the probability that the set of feasible states will expand to $L = \{s_1, s_2, s_3\}$, p, is less than 1, the situation is potentially different. For p sufficiently low, player 1 would be in favor of a switch to s_2, since she would expect that a society will spend a long time in this state. Now suppose that there is an early shock, meaning that s_3 becomes available at time $t = 0$. This early shock make the entire reform process more difficult and discourages player 1 from accepting the change to s_2. In terms of the discussion of democracy in the British context in the Introduction, an early shock would correspond to the elite believing in 1832 that there will be very rapid reform towards much more inclusive franchise immediately. If many members of the elite supported the reforms of 1832 with the understanding that these would be relatively stable, such a shock may have made them less willing to accept the more modest reforms of 1832 in the first place.

Example 2 ***(Patience may make matters worse)*** Consider the same environment as in the previous example, but with $L_0 = \{s_1, s_2, s_3\}$, so that there is no room for new states becoming available. When β is close to 1, player 1 does not wish to accept the eventual switch to s_3, and thus state s_1 will persist. But if β is sufficiently small, then the transitory gain from the change to s_2 is sufficient to compensate player 1 for the lower utility from s_3. In this example, when players are more forward-looking and more patient, the reform becomes more difficult.

This example also illustrates that the process of social evolution implied by our model need not be Pareto efficient. When the discount factor β is close to 1, the unique equilibrium involves the society remaining in state s_1 even though all agents are strictly better off in state s_2. The reason why such Pareto improving change does not take place is that once this transition is implemented, political power shifts away from player 1 to player 2, who will then have an incentive to implement a further change to state s_3, which is disliked by player 1. Thus, it is the inability of player 2 to commit to not implementing deferred or changed that leads to the Pareto efficient outcome.

Example 3 ***(Shocks may make reform easier)*** We now illustrate how, in contrast to Example 1, shocks may also make reform easier. Suppose that $S = \{s_1, s_2, s_3, s_4\}$ and we start with the set of feasible states, $L_0 = \{s_1, s_2, s_3\}$. Suppose also that $M_2 = \{2\}$ and $M_3 = M_4 = \{3\}$. Finally, suppose that $u_2(s_3) > u_2(s_2) > u_2(s_1) > u_2(s_4)$ and $u_1(s_4) < u_1(s_3) < u_1(s_1) < u_1(s_2)$, and $u_3(s_4) > u_3(s_3) > u_3(s_2) > u_3(s_1)$. One interpretation is that s_4 is an extreme anti-aristocratic regime, for example corresponding to communism or some form of radical populism. This is disliked both by player 1 and player 2. Initially, state s_4 is not feasible. Suppose that it becomes feasible with some small probability p at each date (regardless of the current state). If p is sufficiently small, player 2 will opt to move from state s_2 to state s_3 (reasoning that s_4 is unlikely to become feasible and thus s_3 is likely to persist). But since s_3 is disliked by player 1, this means that player 1 prefers to stay with s_1. However, when state s_4 becomes available, player 2 will no longer choose to transition from s_2 to s_3. Given this, player 1 would be happy to transition to s_2. In this case, the shock that made state s_4 available, for example, the organization of a strong socialist or populist party, will make the initial step of reform more likely.

Example 4 ***(The order of shocks matters for the reform process)*** Suppose again that $S = \{s_1, s_2, s_3, s_4\}$, but we now start with the set of feasible states, $L_0 = \{s_2, s_3\}$, and current state s_2 with $M_2 = \{2\}$. We can interpret this as the society starting with a weak monarchy, dominated by the aristocracy, and the only feasible transition is to constitutional monarchy; a transition to either absolutist monarchy or full democracy are not feasible at first. We again have $M_3 = M_4 = \{3\}$. The payoffs are $u_2(s_1) > u_2(s_3) > u_2(s_2) > u_2(s_4)$ and $u_1(s_1) > u_1(s_2) > u_1(s_3) > u_1(s_4)$ and $u_3(s_4) > u_3(s_3) > u_3(s_2) > u_3(s_1)$. Both s_1 and s_4 become available with probability p at each date. Now for p sufficiently small and β sufficiently small, if neither s_1 nor s_4 is available at the first date, player 2 will transition to s_3. If s_1 becomes available at the first date, then player 2 will transition to s_1 immediately. If s_4 becomes available at the first date, then player 2 will not want to transition to s_3, and will wait for s_1 to become available. Thus the exact timing of different types of shocks will determine both the timing of transitions and the ultimate limiting state.